A Primer on LC Resonance and Importance of High-Q Elements in an LC-Oscillator

Presentation · February 2008

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A Primer on *LC* Resonance and Importance of High-*Q* Elements in an *LC*-Oscillator

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February 5, 2008

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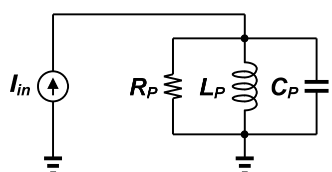
Introduction

To help understand how *LC* resonant oscillators work and why we desire to have high-*Q* inductors and capacitors, let's first review some resonant circuit concepts from freshman college physics (or high school physics for keeners). No denying it's really basic stuff but some of the basic goodies often get overlooked. Enjoy.

What is Parallel Resonance?

Parallel resonance occurs when the *admittance* of a parallel *RLC* network becomes purely real.



The admittance of a parallel *RLC* network is given by

$$Y_P(\omega) = \frac{1}{R_P} + \frac{1}{j\omega L_P} + j\omega C_P = \frac{1}{R_P} + j\left(\omega C_P - \frac{1}{\omega L_P}\right) \tag{1}$$

To calculate when parallel resonance occurs, we set the susceptance (imaginary component of admittance) to zero and not forgetting that $\frac{1}{j} = -j$ (or $j^2 = -1$) ☺, we get

$$\Im\left[Y_P(\omega_P)\right] = 0 \Rightarrow \omega_P C_P - \frac{1}{\omega_P L_P} = 0 \tag{2}$$

and arrive at the very familiar result

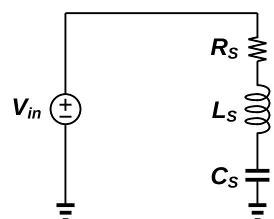
$$\omega_P = \frac{1}{\sqrt{L_P C_P}}. \tag{3}$$

When parallel resonance occurs, we *maximize the AC voltage* across the *RLC* network for a *given AC current*. Since the admittance is purely real at resonance, the AC voltage is exactly in phase with the external AC current.

Physically, energy storage is sloshing back and forth at the resonance frequency between the capacitor's electric fields (as $\frac{1}{2}CV^2$ when voltage appears across the capacitor) and the inductor's magnetic fields (as $\frac{1}{2}LI^2$ when current flows through the inductor). In the case when R_P is infinite, the shunt resistor disappears and there is no loss mechanism. The resulting total energy stored in both capacitor and inductor remains constant without an external energy source, and hence the voltage amplitude remains constant and persists indefinitely without need to have external energy injected periodically in order to sustain the voltage oscillations. If R_P is not infinite, any voltage across L_P-C_P will cause current to flow through R_P and hence dissipate heat. In this case, external energy must be supplied to overcome the ohmic loss in R_P or the voltage oscillation will decay in amplitude and eventually cease to exist.

What is Series Resonance?

Series resonance occurs when the *impedance* of a series *RLC* network becomes purely real.



The impedance of a series *RLC* network is given by

$$Z_S = R_S + \omega L_S + \frac{1}{j\omega C_S} = R_S + j\left(\omega L_S - \frac{1}{\omega C_S}\right) \tag{4}$$

To calculate when series resonance occurs, we set the reactance (imaginary component of impedance) to zero

$$\Im[Z_S(\omega_S)] = 0 \Rightarrow \omega_S L_S - \frac{1}{\omega_S C_S} = 0 \tag{5}$$

and arrive at the just-as-familiar result

$$\omega_S = \frac{1}{\sqrt{L_S C_S}}. \tag{6}$$

When parallel resonance occurs, we *maximize the AC current* across the *RLC* network for a *given AC voltage*. Since the impedance is purely real at resonance, the AC current is exactly in phase with the external AC voltage.

Dealing with Other *RLC* Networks

It is very critical to realize that *Eqs. (3) and (6) hold ONLY for purely parallel and series RLC networks respectively*. If you have an *RLC* network that is not an exact match to either a parallel or series *RLC* network, you obviously cannot use Eq. (3) or (6) – not terribly profound but a common mistake one may be tempted to make.

To obtain the resonance frequency for a more complex *RLC* network, you have two approaches:

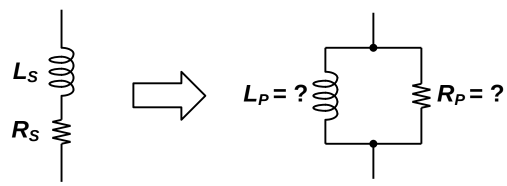
Method 1. Compute the net impedance or admittance, and force the imaginary component to zero to solve for the resonance condition (you quickly rediscover your frustration with algebra this way) , or

Method 2. *Transform* the *RLC* network into a purely parallel *RLC* or series *RLC* network in order to apply Eqs. (3) or (6) using the *transformed* values of *L* and *C*.

Impedance Transformations

So what's an impedance transformation after all? The basic goal is to map an impedance (R+jX) into an equivalent admittance (G+jS), or vice versa.

For starters, take an inductor L_S with some parasitic series resistance R_S. Our goal is to map the series L_S-R_S combination into an equivalent parallel L_P-R_P combination.



We simply write

$$Z_S = j\omega L_S + R_S \quad \text{and} \quad Y_P = \frac{1}{j\omega L_P} + \frac{1}{R_P} \qquad \text{(7a), (7b)}$$

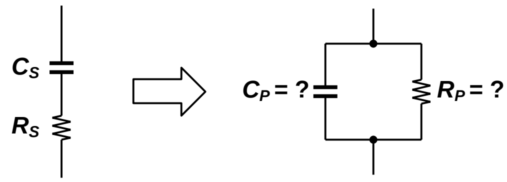
Equating real and imaginary components of $\frac{1}{Z_S} \equiv Y_P$, we obtain after some manipulation

$$L_P = L_S \cdot \left(1 + \frac{R_S^2}{\omega^2 L_S^2}\right) \text{ and} \qquad (8)$$

$$R_P = R_S \cdot \left(1 + \frac{\omega^2 L_S^2}{R_S^2}\right). \qquad (9)$$

Notice that the transformation is *frequency-dependent*, so using fixed values of L_P and R_P is a good approximation only over a narrow frequency band. Also, observe that when the inductor is ideal (R_S = 0), we arrive at the intuitively satisfying result $L_P = L_S$ since R_P is infinite.

We now move on to transform a capacitor C_S with parasitic series resistance R_S.



We write

$$Z_S = \frac{1}{j\omega C_S} + R_S \quad \text{and} \quad Y_P = j\omega C_P + \frac{1}{R_P} \qquad \text{(10a), (10b)}$$

and obtain similar expressions

$$C_P = C_S \cdot \left(\frac{1}{1+\omega^2 C_S^2 R_S^2} \right) \text{ and} \qquad (11)$$

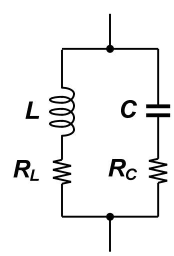
$$R_P = R_S \cdot \left(1 + \frac{1}{\omega^2 C_S^2 R_S^2} \right). \qquad (12)$$

Once again, note the frequency dependence of the transformation. And also note that when the capacitor is ideal ($R_S = 0$), we arrive at the unsurprisingly result $C_P = C_S$ since R_P is infinite.

A Practical Example

You may suspect that the preceding transformation examples, i.e., the fact that I've glibly ignored transforming admittances back to impedances, seem to suggest that parallel *RLC* networks are perhaps more useful than series *RLC* networks, and this is probably true. Why? Well, when we build resonant circuits, voltage is usually the output variable of interest, *not* current. In other words, we're usually in the business of generating a large voltage swing using as little current or power as possible.

Consider the following *RLC* network.



This is a decent representation of a tank network in basic *LC* oscillators. R_L seems representative of metal resistance in a planar spiral inductor while R_C represents series channel and source/drain resistance in a voltage-tunable MOS varactor. After all of the above commotion, we could now conclude confidently and emphatically that the resonant frequency $\omega_{res} = \frac{1}{\sqrt{LC}}$, couldn't we?

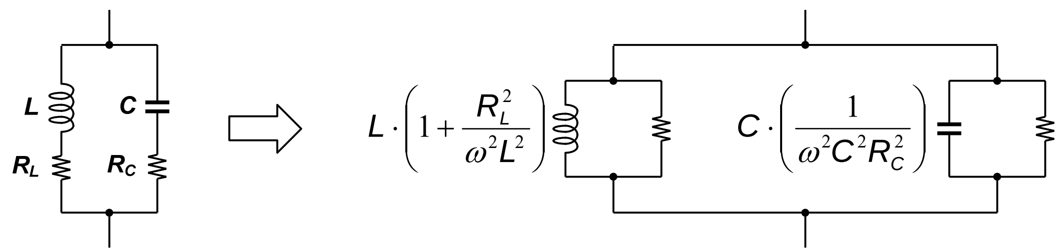
If we proceed with Method 1, we'll quickly arrive at the following expression for net impedance:

$$Z = (j\omega L + R_L) \,||\, \left(\frac{1}{j\omega C} + R_C \right) \tag{13}$$

After making the imaginary component vanish and enduring some algebra, we get

$$\omega_{res} = \frac{1}{\sqrt{LC}} \cdot \sqrt{\frac{1 - \frac{C}{L} R_L^2}{1 - \frac{C}{L} R_C^2}} \tag{14}$$

We can also proceed with Method 2 and arrive at Eq. (14) more quickly using Eqs. (8), (9), (11), and (12).



Grouping the two transformed shunt resistors together, we're back to our faithful parallel *RLC* network. The resulting resonant frequency is hence

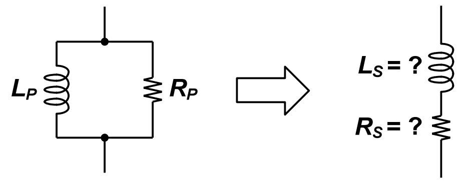
$$\omega_{res} = \frac{1}{\sqrt{L_P C_P}} = \frac{1}{\sqrt{LC}} \cdot \sqrt{\frac{1 + \omega_{res}^2 C^2 R_C^2}{1 + \frac{R_L^2}{\omega_{res}^2 L^2}}} \tag{15}$$

I'll leave it as an exercise for you, the reader, to convince yourself that Eqs. (14) and (15) are indeed equivalent. ☺

Regardless of which method you choose to compute the resonant frequency, it should be clear that $\omega_{res} = \frac{1}{\sqrt{LC}}$ only when the series losses are nulled out but when is reality ever this ideal?

Admittance Transformations

So I've lied a little by understating the transformation of admittances back to impedances. For completeness, let's transform a shunt inductance/resistance combination.

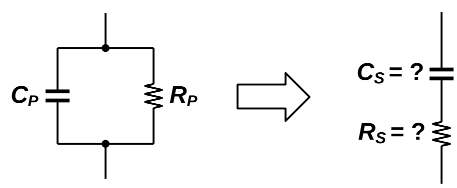


With a little math, we readily obtain

$$L_S = L_P \cdot \left(\frac{1}{1 + \frac{\omega^2 L_P^2}{R_P^2}} \right) \text{ and} \tag{16}$$

$$R_S = R_P \cdot \left(\frac{\frac{\omega^2 L_P^2}{R_P^2}}{1 + \frac{\omega^2 L_P^2}{R_P^2}} \right). \tag{17}$$

For a shunt capacitance/resistance combination,

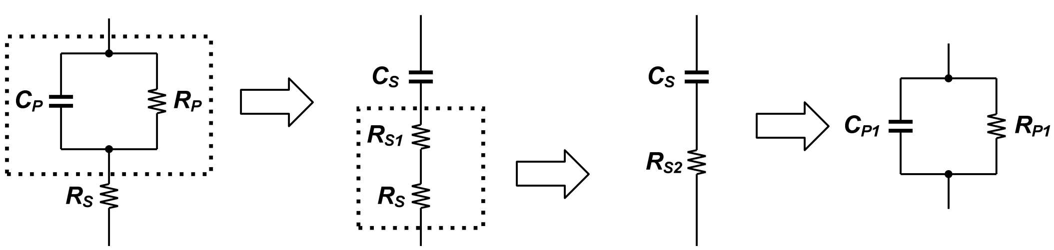


we obtain similar equations

$$C_S = C_P \cdot \left(\frac{1 + \omega^2 C_P^2 R_P^2}{\omega^2 C_P^2 R_P^2} \right) \text{ and} \tag{18}$$

$$R_S = R_P \cdot \left(\frac{1}{1 + \omega^2 C_P^2 R_P^2} \right). \tag{19}$$

The combination of impedance and admittance transformation equations could prove useful, for example, to reduce the following more realistic model of a MOS varactor, where R_P represents gate leakage, into a simple shunt capacitance/resistance network.



Why the Fuss about *Q*'s?

When we think of the Q or *quality factor* of a resonant network, we typically think of the ratio of resonant frequency to bandwidth, i.e., the frequency selectivity sharpness of a tank's bandpass filter action. However, the interpretation of inductor or capacitor Q may not be so obvious. Fundamentally,

$$Q \equiv \omega \cdot \frac{energy.stored}{power.dissipated} \tag{20}$$

It is the ratio of energy stored in the reactive element to the power dissipated in that element's loss mechanism(s). So Q is really a measure of a reactive element's ideality where infinite Q implies a perfectly lossless energy storage element; stated conversely, the lower the Q, the lossier the element.

Sparing the derivation details, Eq. (20) can be employed to derive

$$Q_L \approx \frac{\omega L}{R_L} \text{ and} \tag{21}$$

$$Q_C \approx \frac{1}{\omega C R_C} \ . \tag{22}$$

where R_L and R_C are resistances in series with L and C respectively. Although having only a simple series resistance is a gross lumped modeling oversimplification for integrated inductors and capacitors, Eqs. (17) and (18) do provide a very basic insight. To first order, they suggest that *inductor Q improves with increasing frequency while capacitor Q does the opposite*.

In practical ICs, this means that inductor Q typically limits tank Q at lower frequencies while capacitor Q typically limits tank Q at higher frequencies. So as LC oscillators are built to target even higher frequencies, the importance of high-Q capacitors becomes increasingly higher.

Why Use High-*Q* Elements for *LC* Oscillators?

Simple –.for low power and more importantly, for low noise and low clock jitter. Quieter clocks mean higher operating frequencies because the window of uncertainty for timing events is much tighter.

The effective Q of a tank is computed as

$$\frac{1}{Q_{eff}} = \frac{1}{Q_L} + \frac{1}{Q_C} \ . \tag{23}$$

Like resistors in parallel, the tank Q is limited by the smaller of Q_L and Q_C. If we use high-Q inductors and capacitors, the equivalent shunt resistance of a parallel RLC network will be large. Ohm's Law tells us that a small amount of externally injected current will develop a large voltage oscillation amplitude across the shunt resistor, which intuitively translates to high signal-to-noise ratio. And if we depend on energy from noisy transistors less frequently to sustain oscillation, that ultimately means less $1/f$ and thermal noise being added to the tank.

The Importance of Accurate *LC* Modeling

One of the beauties of *LC* oscillators is that if you model the *L* and *C* elements accurately, the oscillator frequency can be predicted with high accuracy. That simple! Problem is that building high-*Q* *L*'s and *C*'s is not achievable in a monolithically integrated oscillator, which means you need to get R_L and R_C right too. Restating Eq. (14) below, R_L and R_C clearly cannot be ignored.

$$\omega_{res} = \frac{1}{\sqrt{LC}} \cdot \sqrt{\frac{1-\frac{C}{L}R_L^2}{1-\frac{C}{L}R_C^2}} \neq \frac{1}{\sqrt{LC}} \qquad (14)$$

I'll throw out a simple example to illustrate how the impact of R_L might be important. Consider what happens when the temperature of the oscillator increases from say 0°C to 100°C. In a practical oscillator, *L* and *C* will be fairly constant across temperature but the temperature coefficient of R_L could be significant. For example, one would expect R_L to increase quite dramatically since metal resistance typically increases by 30% across 100°C. Having said this, Eq. (14) says the oscillator frequency will decrease as temperature increases. Indeed, this is typically observed in silicon because most practical oscillators are still dominated by R_L instead of R_C. This temperature sensitivity also impacts the tuning range requirement of an oscillator. Since oscillators are most practically used in a phase-locked loop where the output phase (and frequency) is locked to a much lower frequency reference in order to produce a stable clock independent of PVT, the capacitance that tunes the oscillator must clearly be adjusted in order to maintain a constant oscillator frequency as temperature increases. The degree of capacitance adjustment required is obviously a function of how much the oscillator frequency would want to drift across temperature if no capacitance feedback correction were applied. Clearly, none of these considerations would surface if we had only naively considered *L* and *C* while neglecting the impact of R_L and R_C.

References

[1] A. L. S. Loke, R. K. Barnes, T. T. Wee, M. M. Oshima, C. E. Moore, R. R. Kennedy, and M. J. Gilsdorf, “A versatile 90-nm CMOS charge-pump PLL for SerDes transmitter clocking,” *IEEE J. Solid-State Circuits*, vol. 41, no. 8, pp. 1894-1907, Aug. 2006.

[2] *The Design of CMOS Radio-Frequency Integrated Circuits* 2nd ed. by T. H. Lee, Cambridge (2003).

[3] A. Hajimiri and T. H. Lee, “Design issues in CMOS differential *LC* oscillators,” *IEEE J. Solid-State Circuits*, vol. 34, no. 5, pp. 717-724, May 1999.